From risk analysis to adversarial risk analysis

Part III. Modelling preferences under uncertainty. Utilities

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Modeling preferences under uncertainty

Build a function associating a number (utility) to each consequence, so that we compute the expected utility of each action

$$u(a) = \sum_{\theta} u(a,\theta)\, P(\theta)$$

Optimal alternative: **maximum expected alternative**

Which of these lotteries do you prefer?

$$\mathbf{A} = \begin{pmatrix} 1/2 & 1/2 \\ 1000 & -600 \end{pmatrix} \; y \; la \; \mathbf{B} = \begin{pmatrix} 1/2 & 1/2 \\ 100000 & -50000 \end{pmatrix}$$

Modeling preferences under uncertainty

Consider this situation

Throw a (balanced) coin until head appears. If this happens in the n-th draw, you get 2^n euros.

How much are you ready to pay to enter in the game??

Modeling preferences under uncertainty

Expected monetary value is not always an appropriate criteria for decision making under uncertainty

Expected utility provides an appropriate solution (at least normatively)

Modeling preferences under uncertainty

A sketch of formalisation

http://www.econ2.jhu.edu/people/karni/chapter.pdf

Provides a rigorous description

Modeling preferences under uncertainty

DM expresses preferences over lotteries

- Preferences rational: transitive and complete
- Independence property
- Continuity property (no heaven, no hell)

Preferences modeled through expected utility

$$\max_a \int u(c(a,\theta))p(\theta|x)d\theta$$

Utility is affine unique

Modeling preference. Single criteria

- Determine range of interest
- Assign utility 0 to worst value and utility 1 to best value
- Assign utilities to a few intermediate values (eg EP)
- Fit a utility function (e.g. through nonlinear least squares)
- Check for consistency
- Perform sensitivity analysis

- Take into account biases
- It. Imprecision

Details

http://www.jstor.org/pss/2631564

Modelling preferences. Risk attitudes

- Three basic risk attitudes
  - Risk aversion. Concave utility
  - Risk proneness. Convex utility
  - Risk neutrality. Linear utility

Consequence varying risk attitudes

Risk attitudes

Compare lotteries A and B. Which one do you prefer?

$$\mathrm{A} = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 100000 \end{pmatrix} \text{ y } \mathrm{B} = \begin{pmatrix} 1 \\ 50000 \end{pmatrix}.$$

Risk attitudes

Compare lotteries A and B. Which one do you prefer?

$$A = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 100000 \end{pmatrix} \text{ y } B = \begin{pmatrix} 1 \\ 50000 \end{pmatrix}.$$

If you prefer B to A, you seem to prefer the sure prize to the lottery (risk aversion)

In such case, the expected utility of B should be bigger than that of A:

$\frac{1}{2} * u(0) + \frac{1}{2} * u(100000) = \frac{1}{2} < 1 * u(50000)$

Graphically…

Risk attitudes

Compare lotteries A and B. Which one do you prefer?

$$A = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 100000 \end{pmatrix} \text{ y } B = \begin{pmatrix} 1 \\ 50000 \end{pmatrix}.$$

If you prefer A to B, you seem to prefer the lottery to the sure prize (risk proneness)

In such case, the expected utility of A should be bigger than that of B:

$½ * u(0) + ½ * u(100000) = ½ > 1 * u(50000)$

Graphically…

Risk attitudes

Compare lotteries A and B. Which one do you prefer?

$$A = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 100000 \end{pmatrix} \text{ y } B = \begin{pmatrix} 1 \\ 50000 \end{pmatrix}.$$

If you are indifferent between A and B, you seem indifferent between the lottery and the sure prize (risk neutrality)

In such case, the expected utilities of A and B should be equal:

$½ * u(0) + ½ * u(100000) = ½ = 1 * u(50000)$

Graphically…

Risk attitudes

Formally, for a lottery

$(p_1,x_1;p_2,x_2;\ldots;p_n,x_n)$

consider its monetary expected value MEV

$p_1*x_1+p_2*x_2+\ldots+p_n*x_n$

and its certainty equivalent CE

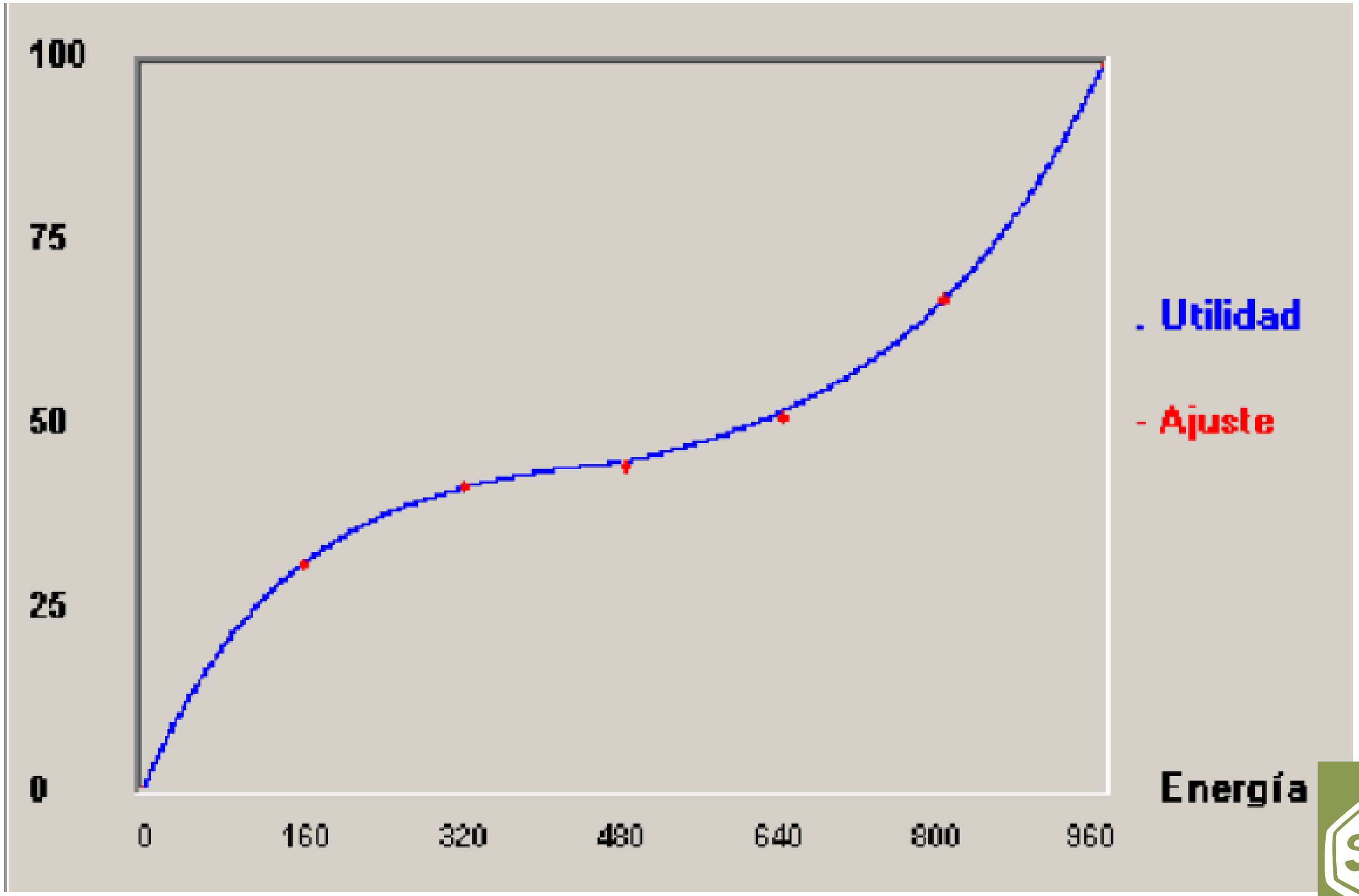
$u^{-1}(p_1*u(x_1)+p_2*u(x_2)+\ldots+p_n*u(x_n))$

- If MEV>CE, risk aversion
- If MEV<CE, risk proneness
- If MEV=CE, risk neutrality

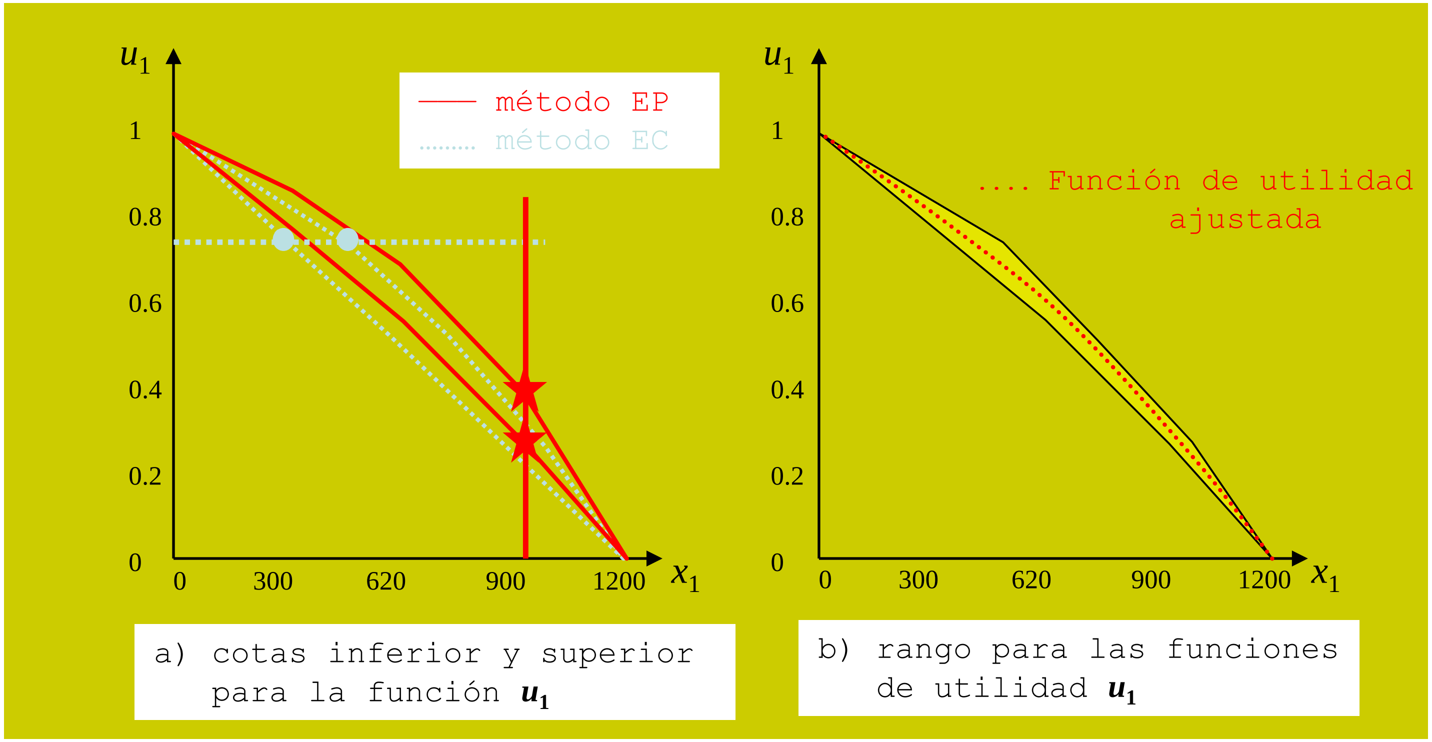
Modelling preferences. Risk Aversion

- Risk premium=CE-EMV
- Absolute Risk Aversion. $-u''(c)/u'(c)$
  -
- Constant Absolute Risk Aversion. Exponential utility function.

Which risk attitude??



Which risk attitude? (Care it is decreasing)



Multicriteria preferences under uncertainty

- Determine preference independence conditions
- Assess utility components
- Assess weights

As before

- Evaluate consistency of assignment
- Perform sensitivity analysis

Taking into account

- Biases
- Imprecision

Multicriteria preferences under uncertainty

Utility function additive if

$$u(x, y) = \lambda_1 u_1(x) + \lambda_2 u_2(y),$$

Additive independence condition:

$$\begin{pmatrix} 1/2 & 1/2 \\ (x, y) & (x', y') \end{pmatrix} \sim \begin{pmatrix} 1/2 & 1/2 \\ (x, y') & (x', y) \end{pmatrix} \forall\ x, x', y, y'.$$

To determine λ_2 determine α such that

$$\begin{pmatrix} 1-\alpha & \alpha \\ (x_*, y_*) & (x^*, y^*) \end{pmatrix} \sim (x_*, y^*).$$

Utility functions in a medical problem

$$u_1(x_1) = 1.604 - 0.604\exp(0.00077x_1)$$

$$u_2^x(x_2) = -0.1108 + 1.111\exp(-1.153x_2)$$

$$u_3^x(x_3) = -0.225 + 1.225\exp(-0.8473x_3)$$

$$u_4(x_4) = 1.277 - 0.2766\exp(0.5098x_4)$$

$$u_5^x(x_5) = 1.361 - 0.361\exp(0.3316x_5)$$

$$u_6^x(x_6) = 1.408 - 0.403\exp(0.2476x_6)$$

Logical Decisions